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Extensions of extremum principles for slow viscous flows

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Several generalizations of theorems of the types originally stated by Helmholtz concerning the dissipation of energy in slow viscous flow have been given recently by Keller, Rubenfeld & Molyneux (1967). These generalizations included cases in which the fluid contains one or more solid bodies and drops of another liquid assuming the drops do not change shape. Some further extensions are given herein which allow for drops which may be deformed by the flow and include the effect of surface tension. The admissible boundary conditions have also been extended and particular theorems applicable to infinite domains, spatially periodic flows and to flows in infinite cylindrical pipes are derived. Uniqueness theorems are also proved.

1. Introduction

The history of extremum principles for slow viscous flow (Stokes flow) is given briefly by Keller *et al.* (1967) and they prove several theorems which include and extend previous results. These theorems establish upper and lower bounds for the excess dissipation rate which is defined to be the rate of energy dissipation in the fluid minus twice the power of the external body forces and given surface tractions. One of the principal generalizations introduced was to include suspended solid particles and drops of another liquid whose motion is not known in advance. However, the shapes of the drops were assumed to be constant during the motion.

In the present paper it is shown that if minus twice the power delivered by surface tension is included in the definition of the excess dissipation rate, that minimum and maximum principles can be derived for suspensions containing deformable drops as well as rigid particles.

The motivation for the present paper stems from a study of capillary blood flow in which the red blood cells may be represented by a line of flexible particles suspended in viscous flow in a tube. Spatially periodic flows are of interest for this application and the extremum principles have also been appropriately specialized for this purpose. Simplifications in the specification of the problem are possible, namely, it is sufficient to specify certain integral quantities such as the discharge rather than pointwise data such as velocity on the boundaries of the typical periodic cell. These results can be applied to uniform flows in cylindrical pipes also.

Specific theorems are proved for infinite domains assuming body forces are conservative. It is shown that the rate of decrease of the velocity at infinity can be predicted rather than assumed and this allows a more general statement of uniqueness and extremum principles.

All of the theorems proved below, like those already in the literature, consider that the configurations of the droplets and particles are known at the instant that the fluid motion is to be found.

2. Statement of the problem

Consider a domain V which contains a viscous fluid in which there are N_L liquid drops or bubbles and N_K rigid, solid particles. The boundary S of V is assumed not to intersect any of the suspended drops or particles. The boundary conditions in the sense of prescribed velocity and traction components are assumed to be specified only on S which will be subdivided into S_1, S_2, S_3, S_4 according to the particular components specified.

Let the domain V be subdivided into V_0 occupied by the suspending fluid, $V_L^{(l)}$ $(l = 1, \ldots, N_L)$ occupied by the fluid drops, and $V_K^{(k)}$ $(k = 1, \ldots, N_K)$ occupied by the solid particles. Let S_0, $S_L^{(l)}$ and $S_K^{(k)}$ denote the surfaces of V_0, $V_L^{(l)}$ and $V_K^{(k)}$ respectively. Then S_0 is the sum of S, $S_L^{(l)}$ $(l = 1, \ldots, N_L)$ and $S_K^{(k)}$ $(k = 1, \ldots, N_K)$. Let $\mathbf{n}$ denote the normal to S_0 directed outward from V_0.

Each of the fluids in V_0 and $V_L^{(l)}$ $(l = 1, \ldots, N_L)$ is assumed to be a uniform, incompressible, Newtonian fluid but the viscosity $\mu(\mathbf{x})$ may be different in each of these domains. Let $\sigma^{(l)}$ $(l = 1, \ldots, N_L)$ denote the surface tension in $S_L^{(l)}$; $\sigma^{(l)}$ may be different for each $S_L^{(l)}$.

It is convenient to define a single velocity field $\mathbf{u}(\mathbf{x})$ for the entire domain V. The motions of the drops and solid particles are not known in advance but are to be found as part of the solution. The requirement of zero relative velocity of the fluids and solids on the two sides of each $S_L^{(l)}$ and $S_K^{(k)}$ is met by stipulating that $\mathbf{u}(\mathbf{x})$ be continuous in V. Within each solid particle, the velocity $\mathbf{u}(\mathbf{x})$ is defined to be that of the rigid body motion consistent with the fluid velocity on its boundary.

Let $\mathbf{f}(\mathbf{x})$ denote the body force per unit volume defined throughout V. Let $p(\mathbf{x})$ and $\tau_{ij}(\mathbf{x})$ denote the pressure and stress tensor which are defined only in the fluid domains V_0 and $V_L^{(l)}$ $(l = 1, \ldots, N_L)$. The pressure and stress are required to be continuous except across the surfaces $S_L^{(l)}$ of the drops where the difference of the value outside minus the value inside the drop will be denoted by Δp and $\Delta\tau_{ij}$ respectively.

Let $\mathbf{j}(x)$, $\mathbf{t}(x)$, $\mathbf{m}(x)$ be three unit vectors which are specified at each point of S as part of the boundary conditions of the problem. The $\mathbf{j}$, $\mathbf{t}$, $\mathbf{m}$ must be mutually orthogonal, but may be otherwise arbitrarily oriented at each point.

The problem is to find $\mathbf{u}(\mathbf{x})$ in V satisfying the following equations and boundary conditions:

$$u_{i,i} = 0, \quad \mathbf{x} \quad \text{in} \quad V; \tag{2.1}$$

$$\tau_{ij,j} + f_i = 0, \quad \mathbf{x} \quad \text{in} \quad V_0 \quad \text{and} \quad V_L^{(l)} \quad (l = 1, \ldots, N_L); \tag{2.2}$$

$$u_i = g_i(\mathbf{x}), \quad \mathbf{x} \quad \text{on} \quad S_1; \tag{2.3}$$

$$u_i j_i = h(\mathbf{x}), \quad u_i t_i = b(\mathbf{x}), \quad \mathbf{x} \text{ on } S_2; \tag{2.4a}$$

$$\tau_{ij} n_j m_i = \beta(\mathbf{x}), \quad \mathbf{x} \text{ on } S_2; \tag{2.4b}$$

$$u_i j_i = h(\mathbf{x}), \quad \mathbf{x} \text{ on } S_3; \tag{2.5a}$$

$$\tau_{ij} n_j t_i = \alpha(\mathbf{x}), \quad \tau_{ij} n_j m_i = \beta(\mathbf{x}), \quad \mathbf{x} \text{ on } S_3; \tag{2.5b}$$

$$\tau_{ij} n_j = \gamma_i(\mathbf{x}), \quad \mathbf{x} \text{ on } S_4; \tag{2.6}$$

$$\Delta\tau_{ij} n_i n_j = \sigma^{(l)}\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \quad \mathbf{x} \text{ on } S_L^{(l)} \quad (l = 1, \ldots, N_L); \tag{2.7}$$

$$\Delta\tau_{ij} n_j - \Delta\tau_{qm} n_q n_m n_i = 0, \quad \mathbf{x} \text{ on } S_L^{(l)} \quad (l = 1, \ldots, N_L); \tag{2.8}$$

$$\int_{V_K^{(k)}} f_i dV - \int_{S_K^{(k)}} n_j \tau_{ij} dS = 0, \quad (k = 1, \ldots, N_K); \tag{2.9}$$

$$\int_{V_K^{(k)}} \epsilon_{ijm} x_j f_m dV - \int_{S_K^{(k)}} \epsilon_{ijm} x_j n_q \tau_{qm} = 0 \quad (k = 1, \ldots, N_K); \tag{2.10}$$

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij}, \quad \mathbf{x} \text{ in } V_0 \text{ and } V_L^{(l)} \quad (l = 1, \ldots, N_L); \tag{2.11}$$

$$e_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \quad \mathbf{x} \text{ in } V; \tag{2.12}$$

$$e_{ij} = 0, \quad \mathbf{x} \text{ in } V_K^{(k)} \quad (k = 1, \ldots, N_K); \tag{2.13}$$

where ϵ_{ijk} is the alternating tensor and δ_{ij} is the Kronecker delta. R_1 and R_2 denote the two principal radii of curvature of $S_L^{(l)}$ reckoned positive when they extend into the drop. The functions α, β, γ, g, h, b are given as part of the boundary conditions.

Equations (2.1) and (2.2) are the equations of continuity and motion for the Stokes flow in the fluid domains.

The boundary conditions (2.3), (2.4*a*), (2.5*a*) and (2.6) specify 3, 2, 1 or 0 components of the velocity on S_1, S_2, S_3, S_4 respectively. In each case, a sufficient number of traction components are also specified by (2.4*b*), (2.5*b*) and (2.6) to make the solution unique, as will be shown in the derivations below.

Equation (2.7) equates the difference of the normal components of the tractions on the two sides of $S_L^{(l)}$ to the effect of surface tension $\sigma^{(l)}$. Equation (2.8) states that the tangential component of the surface traction is continuous across $S_L^{(l)}$.

The equations of motion of the solid particles are expressed by (2.9) and (2.10).

Equations (2.11) and (2.12) define the stress tensor τ_{ij} and e_{ij} for a Newtonian fluid and (2.13) ensures that the motion within $S_K^{(k)}$ is that of a rigid body.

Only solutions $\mathbf{u}(\mathbf{x})$ which are continuous throughout V will be considered; derivatives of $\mathbf{u}$ may be discontinuous on $S_L^{(l)}$ $(l = 1, \ldots, N_L)$ and on $S_K^{(k)}$ $(k = 1, \ldots, N_K)$.

The domain V is considered to be finite until §6 where infinite domains are specifically considered. Spatially periodic flows are treated in §7.

3. A minimum principle

Let the rate of dissipation of energy into heat by viscosity in V be denoted $D[\mathbf{u}]$. It is defined by

$$D[\mathbf{u}] = \int_{V_0} 2\mu(e_{ij}[\mathbf{u}])^2 dV + \sum_{l=1}^{N_L} \int_{V_L^{(l)}} 2\mu(e_{ij}[\mathbf{u}])^2 dV. \quad (3.1)$$

The excess dissipation rate $D_e[\mathbf{u}]$ is defined to be the rate of viscous energy dissipation minus twice the power of the external body forces, the given surface traction components and the surface tensions:

$$D_e[\mathbf{u}] = D[\mathbf{u}] - 2\int_V f_i u_i dV - 2\int_{S_2} u_i m_i \beta dS - 2\int_{S_3} (u_i t_i \alpha + u_i m_i \beta) dS - 2\int_{S_4} u_i \gamma_i dS + 2\sum_{l=1}^{N_L} \sigma^{(l)} \dot{A}^{(l)}, \quad (3.2)$$

where $\dot{A}^{(l)}$ is the time rate of change of the area $A^{(l)}$ of $S_L^{(l)}$ $(l = 1, \ldots, N_L)$. The product $(-\sigma^{(l)}\dot{A}^{(l)})$ is the rate at which surface tension does work on the adjacent fluids and is also the rate at which the surface energy $\sigma^{(l)}A^{(l)}$ decreases. At any time, $\dot{A}^{(l)}$ is given by (cf. Landau & Lifshitz 1959)

$$\dot{A}^{(l)} = -\int_{S_L^{(l)}} u_i n_i \left(\frac{1}{R_1} + \frac{1}{R_2}\right) dS. \quad (3.3)$$

The minus sign in (3.3) is due to the fact that $\mathbf{n}$ is the normal taken outward from V_0 which is inward to $V_L^{(l)}$.

THEOREM 1. *A minimum principle. Let* $\mathbf{u}(\mathbf{x})$ *be a continuous solution of a Stokes flow problem satisfying* (2.1)–(2.13). *Let* $\bar{\mathbf{u}}(\mathbf{x})$ *be any continuous velocity field which is piecewise continuously differentiable and satisfies* (2.1), (2.3), (2.4*a*), (2.5*a*), *and* (2.13). *Then*

$$D_e[\mathbf{u}] \leqslant D_e[\bar{\mathbf{u}}] \quad (3.4)$$

The equality holds only if $\bar{\mathbf{u}} = \mathbf{u}$ or $\bar{\mathbf{u}} = \mathbf{u} + \mathbf{u}_0$ where $\mathbf{u}_0$ is a rigid body motion.

(Note that the configurations of the droplets and solid particles are identical for both flows $\mathbf{u}$ and $\bar{\mathbf{u}}$ at the instant considered.)

Proof. Let $\bar{\mathbf{u}} = \mathbf{u} + \tilde{\mathbf{u}}$. Then from (3.1)

$$D[\bar{\mathbf{u}}] = D[\mathbf{u} + \tilde{\mathbf{u}}] = D[\mathbf{u}] + D[\tilde{\mathbf{u}}] + \int_{V_0} 4\mu e_{ij}[\mathbf{u}] e_{ij}[\tilde{\mathbf{u}}] dV + \sum_{l=1}^{N_L} \int_{V_L^{(l)}} 4\mu e_{ij}[\mathbf{u}] e_{ij}[\tilde{\mathbf{u}}] dV. \quad (3.5)$$

In (3.5), $2\mu e_{ij}[\mathbf{u}]$ may be replaced by $\tau_{ij}[\mathbf{u}]$ because the trace of $e_{ij}[\tilde{\mathbf{u}}]$ is zero. Also, $e_{ij}[\tilde{\mathbf{u}}]$ may be replaced by $\tilde{u}_{i,j}$ because $\tau_{ij}[\mathbf{u}]$ is symmetric. Then using (2.2), (3.5) becomes

$$D[\bar{\mathbf{u}}] = D[\mathbf{u}] + D[\tilde{\mathbf{u}}] + \int_{V_0} (2(\partial_j \tilde{u}_i \tau_{ij}[\mathbf{u}]) + 2f_i \tilde{u}_i) dV + \sum_{l=1}^{N_L} \int_{V_L^{(l)}} (2\partial_j(\tilde{u}_i \tau_{ij}[\mathbf{u}]) + 2f_i \tilde{u}_i) dV. \quad (3.6)$$

Now replacing $\mathbf{u}$ in (3.2) by $\bar{\mathbf{u}}$ and using (3.6) and Gauss's theorem, yields (3.7), below. In applying Gauss's theorem the surfaces on which the derivatives of $\bar{\mathbf{u}}$ are discontinuous give no net contributions. The contributions from the two sides cancel because $\bar{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ are continuous. Thus

$$
\begin{aligned}
D_e[\bar{\mathbf{u}}] = {} & D[\mathbf{u}]+D[\tilde{\mathbf{u}}]+2\int_{S_0}\tilde{u}_i\tau_{ij}^{(0)}[\mathbf{u}]n_j\,dS+2\int_{V_0}f_i\tilde{u}_i\,dV \\
& -2\sum_{l=1}^{N_L}\int_{S_L^{(l)}}\tilde{u}_i\tau_{ij}^{(l)}[\mathbf{u}]n_j\,dS+2\sum_{l=1}^{N_L}\int_{V_L^{(l)}}f_i\tilde{u}_i\,dV \\
& -2\int_V f_i(u_i+\tilde{u}_i)\,dV-2\int_{S_2}(u_i+\tilde{u}_i)m_i\beta\,dS \\
& -2\int_{S_3}(u_i+\tilde{u}_i)(t_i\alpha+m_i\beta)\,dS-2\int_{S_4}(u_i+\tilde{u}_i)\gamma_i\,dS \\
& +2\sum_{l=1}^{N_L}\sigma^{(l)}(\dot{A}^{(l)}+\dot{\tilde{A}}^{(l)}),
\end{aligned} \tag{3.7}
$$

where $\dot{A}^{(l)}$ is given by (3.3) and $\dot{\tilde{A}}^{(l)}$ is the rate of change of $A^{(l)}$ under the velocity $\tilde{\mathbf{u}}$. Since (3.3) is linear in $\mathbf{u}$, $\dot{\tilde{A}}^{(l)}$ is given by (3.3) with $\mathbf{u}$ replaced by $\tilde{\mathbf{u}}$. In (3.7) the superscripts in $\tau_{ij}^{(0)}[\mathbf{u}]$ and $\tau_{ij}^{(l)}[\mathbf{u}]$ have been added to denote the stress tensors on the two sides of the surfaces $S_L^{(l)}$ facing V_0 and $V_L^{(l)}$ respectively. The difference $(\tau_{ij}^{(0)}[\mathbf{u}]-\tau_{ij}^{(l)}[\mathbf{u}])$ is $\Delta\tau_{ij}$ as used in (2.7) and (2.8).

Since $\mathbf{u}$ and $\bar{\mathbf{u}}$ both satisfy (2.3), (2.4*a*) and 2.5*a*), the components of $\tilde{\mathbf{u}}$ corresponding to the specified components of $\mathbf{u}$ on S are zero. Further, $\tau_{ij}[\mathbf{u}]$ satisfies (2.4*b*), (2.5*b*) and (2.6). As a result, the surface integrals in (3.7) over S_1, S_2, S_3 and S_4 involving $\tilde{\mathbf{u}}$ all cancel. The surviving terms of (3.7) may be written

$$
\begin{aligned}
D_e[\bar{\mathbf{u}}] = {} & D_e[\mathbf{u}]+D[\tilde{\mathbf{u}}]+2\sum_{l=1}^{N_L}\int_{S_L^{(l)}}\tilde{u}_i\Delta\tau_{ij}n_j\,dS \\
& -2\sum_{k=1}^{N_K}\int_{V_K^{(k)}}\tilde{u}_if_i\,dV+2\sum_{k=1}^{N_K}\int_{S_K^{(k)}}\tilde{u}_i\tau_{ij}n_j\,dS+2\sum_{l=1}^{N_L}\sigma^{(l)}\dot{\tilde{A}}^{(l)}.
\end{aligned} \tag{3.8}
$$

The integrals over $V_K^{(k)}$ and $S_K^{(k)}$ in (3.8) are the rate of work done on the solid particles by the body forces f_i and surface tractions $\tau_{ij}[\mathbf{u}]$ under the motion $\tilde{\mathbf{u}}$. Since $\tilde{\mathbf{u}}$ is a rigid body motion within each $V_K^{(k)}$ it has the form

$$
\tilde{u}_i = \tilde{u}_i^{(k)}+\epsilon_{ilm}(\tfrac{1}{2}\epsilon_{ljk}\tilde{u}_{k,j})x_m, \quad \mathbf{x} \quad \text{in} \quad V_K^{(k)}, \tag{3.9}
$$

where $\tilde{\mathbf{u}}^{(k)}$ is a constant vector and the angular velocity $(\frac{1}{2}\epsilon_{ljk}\tilde{u}_{k,j})$ which appears in (3.9) is also constant within $V_K^{(k)}$. Using (3.9), (2.9) and (2.10) it follows that the sum of the integrals over $V_K^{(k)}$ and $S_K^{(k)}$ is zero in (3.8).

The surface integrals over $S_L^{(l)}$ and the surface tension terms in (3.8) may be rewritten using (3.3), (2.7) and (2.8) as

$$
\begin{aligned}
& 2\sum_{l=1}^{N_L}\int_{S_L^{(l)}}\tilde{u}_i\Delta\tau_{ij}n_j\,dS+2\sum_{l=1}^{N_L}\sigma^{(l)}\dot{\tilde{A}}^{(l)} \\
& \qquad = 2\sum_{l=1}^{N_L}\int_{S_L^{(l)}}\tilde{u}_in_i\left\{\Delta\tau_{qm}n_qn_m-\sigma^{(l)}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right\}dS,
\end{aligned} \tag{3.10}
$$

where $\tilde{u}_i \Delta\tau_{ij} n_j$ has been replaced by $\tilde{u}_i n_i \Delta\tau_{qm} n_q n_m$ because the tangential component of $\Delta\tau_{ij} n_j$ is zero by (2.8). The last integrand in (3.10) is zero by (2.7). Hence (3.8) becomes

$$D_e[\bar{\mathbf{u}}] = D_e[\mathbf{u}] + D[\tilde{\mathbf{u}}]. \tag{3.11}$$

Since $D[\tilde{\mathbf{u}}]$ is never negative and is zero only if $\tilde{\mathbf{u}} = 0$ or if $\tilde{\mathbf{u}}$ is a rigid body motion, theorem 1 follows.

If the boundary conditions are such that no rigid body motion is possible satisfying (2.3), (2.4*a*) and (2.5*a*) when the given functions g, h, b are replaced by zeros, then $\tilde{\mathbf{u}}$ cannot be a rigid body motion. In this case $D_e[\bar{\mathbf{u}}] = D_e[\mathbf{u}]$ only if $\bar{\mathbf{u}} = \mathbf{u}$. This is the case, for example, if S_1 contains at least three non-colinear points.

4. A maximum principle

A maximum principle for the Stokes flow problem stated in § 2 can be obtained in terms of a functional $H[\tau_{ij}]$ of the stress tensor τ_{ij}. This functional will be called the excess power. It is defined as twice the power delivered by surface tractions on S acting through the given velocity components g, b, h minus the dissipation expressed in terms of the stress:

$$\begin{aligned} H[\tau_{ij}] = {} & 2\int_{S_1} g_i \tau_{ij} n_j dS + 2\int_{S_2} (hj_i \tau_{ij} n_j + bt_i \tau_{ij} n_j) dS \\ & + 2\int_{S_3} hj_i \tau_{ij} n_j dS - \int_{V_0} \frac{1}{2\mu} (\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV \\ & - \sum_{l=1}^{N_L} \int_{V_L^{(l)}} \frac{1}{2\mu} (\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV. \end{aligned} \tag{4.1}$$

When τ_{ij} is the stress tensor corresponding to a solution $\mathbf{u}$ of (2.1)–(2.13), then

$$H[\tau_{ij}] = D_e[\mathbf{u}]. \tag{4.2}$$

To prove (4.2), consider first that for a solution $\tau_{ij}[\mathbf{u}]$, the volume integrals over V_0 and $V_L^{(l)}$ in (4.1) become equal to those in (3.1) and add up to $D[\mathbf{u}]$. Next, by use of the boundary conditions (2.3)–(2.6) the surface integrals over S_1, S_2 and S_3 in (4.1) may be written:

$$\int_{S_1} g_i \tau_{ij} n_j dS = \int_{S_1} u_i \tau_{ij} n_j dS, \tag{4.3}$$

$$\int_{S_2} (hj_i \tau_{ij} n_j + bt_i \tau_{ij} n_j) dS = \int_{S_2} u_i \tau_{ij} n_j dS - \int_{S_2} u_i m_i \beta dS, \tag{4.4}$$

$$\int_{S_3} hj_i \tau_{ij} n_j dS = \int_{S_3} u_i \tau_{ij} n_j dS - \int_{S_3} (u_i t_i \alpha + u_i m_i \beta) dS. \tag{4.5}$$

Using (4.3)–(4.5) in (4.1) and adding and subtracting twice the integral of $u_i \tau_{ij} n_j$ over S_4 gives

$$\begin{aligned} H[\tau_{ij}] = {} & 2\int_S u_i \tau_{ij} n_j dS - 2\int_{S_2} u_i m_i \beta dS \\ & - 2\int_{S_3} (u_i t_i \alpha + u_i m_i \beta) dS - 2\int_{S_4} u_i \gamma_i dS - D[\mathbf{u}]. \end{aligned} \tag{4.6}$$

The first integral in (4.6) is the rate at which surface tractions on S do work on the fluid in V. This integral may be evaluated by expressing conservation of energy in the form

$$\int_S u_i\tau_{ij}n_j dS + \int_V f_i u_i dV = D[\mathbf{u}] + \sum_{l=1}^{N_L} \sigma^{(l)}\dot{A}^{(l)}. \tag{4.7}$$

Equation (4.7) is not an independent postulate here since it may be shown to follow from (2.1) to (2.13). Using (4.7) in (4.6) to eliminate the integral of $u_i\tau_{ij}n_j$ over S gives $H[\tau_{ij}]$ in a form which is identical to (3.2) so (4.2) is proved.

THEOREM 2. *A maximum principle. Let* $\mathbf{u}(\mathbf{x})$ *be a continuous solution of a Stokes flow problem satisfying* (2.1)–(2.13). *Let* $\bar{\tau}_{ij}$ *be any stress tensor defined in* V_0 *and* $V_L^{(l)}$ *which is piecewise continuous and piecewise continuously differentiable and satisfies* (2.2), (2.4*b*), (2.5*b*), (2.6), (2.7), (2.8), (2.9) *and* (2.10); *on surfaces of discontinuity of* $\bar{\tau}_{ij}$ *the traction* $n_i'\bar{\tau}_{ij}$ *is required to be continuous where* n_i' *is the normal to the surface of discontinuity of* $\bar{\tau}_{ij}$ (*other than the surfaces* $S_L^{(l)}$ *of the drops*). *Then*

$$D_e[\mathbf{u}] \geqslant H[\bar{\tau}_{ij}]. \tag{4.8}$$

The equality in (4.8) holds only if $\bar{\tau}_{ij} = \tau_{ij}$ or $\bar{\tau}_{ij} = \tau_{ij} + p_0\delta_{ij}$ where p_0 is a constant.

Proof. Let $\bar{\tau}_{ij} = \tau_{ij} + \tilde{\tau}_{ij}$, where τ_{ij} is the stress tensor corresponding to the solution $\mathbf{u}$. In (4.8), $H[\bar{\tau}_{ij}]$ is given by (4.1) with τ_{ij} replaced by $\bar{\tau}_{ij}$:

$$\begin{aligned} H[\bar{\tau}_{ij}] = {} & 2\int_{S_1} g_i\bar{\tau}_{ij}n_j dS + 2\int_{S_2} (hj_i\bar{\tau}_{ij}n_j + bt_i\bar{\tau}_{ij}n_j)dS \\ & + 2\int_{S_3} hj_i\bar{\tau}_{ij}n_j dS - \int_{V_0} \frac{1}{2\mu}(\bar{\tau}_{ij} - \tfrac{1}{3}\bar{\tau}_{kk}\delta_{ij})^2 dV \\ & - \sum_{l=1}^{N_L}\int_{V_L^{(l)}} \frac{1}{2\mu}(\bar{\tau}_{ij} - \tfrac{1}{3}\bar{\tau}_{kk}\delta_{ij})^2 dV. \end{aligned} \tag{4.9}$$

The integral over V_0 in (4.9) is

$$\begin{aligned} \int_{V_0} \frac{1}{2\mu}(\bar{\tau}_{ij} - \tfrac{1}{3}\bar{\tau}_{kk}\delta_{ij})^2 dV = {} & \int_{V_0} \frac{1}{2\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV \\ & + \int_{V_0} \frac{1}{\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})dV \\ & + \int_{V_0} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV. \end{aligned} \tag{4.10}$$

An expansion similar to (4.10) can be written for the integrals over $V_L^{(l)}$ in (4.9). Using these expansions in (4.9) and comparing to (4.1) gives

$$\begin{aligned} H[\bar{\tau}_{ij}] = H[\tau_{ij}] & - \int_{V_0} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ & - \sum_{l=1}^{NL}\int_{V_L^{(l)}} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ & - 2\int_{V_0} e_{ij}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})dV \end{aligned}$$

$$-2\sum_{l=1}^{N_L}\int_{V_L^{(l)}} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})\,dV$$

$$+2\int_{S_1} g_i\tilde{\tau}_{ij}n_j dS+2\int_{S_2}(hj_i\tilde{\tau}_{ij}n_j+bt_i\tilde{\tau}_{ij}n_j)\,dS$$

$$+2\int_{S_3} hj_i\tilde{\tau}_{ij}n_j dS. \tag{4.11}$$

In (4.11), (2.11) has been used to replace τ_{ij} in terms of e_{ij}. The integral over V_0 containing e_{ij} in (4.11) may be written

$$\int_{V_0} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})dV=\int_{V_0} e_{ij}\tilde{\tau}_{ij}dV=\int_{V_0} u_{i,j}\tilde{\tau}_{ij}dV$$

$$=\int_{V_0}\partial_j(u_i\tilde{\tau}_{ij})\,dV=\int_{S_0} u_i\tilde{\tau}_{ij}n_j dS. \tag{4.12}$$

In deriving (4.12), account is taken of the facts that u_i is an incompressible flow and that $\partial_j(\tilde{\tau}_{ij})=0$, since both τ_{ij} and $\tilde{\tau}_{ij}$ satisfy (2.2). The surfaces of discontinuity of $\tilde{\tau}_{ij}$ would also enter in (4.12) but since $\tilde{\tau}_{ij}n_j'$ and $\tau_{ij}n_j'$ are both continuous across such surfaces, the contributions over the two sides of these surfaces cancel.

A transformation similar to (4.12) yields

$$\int_{V_L^{(l)}} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})\,dV=-\int_{S_L^{(l)}} u_i\tilde{\tau}_{ij}n_j dS, \tag{4.13}$$

where n_j is again the normal outward from V_0. The surface S_0 in (4.12) is the sum of S_1, S_2, S_3, S_4, $S_L^{(l)}$ and $S_K^{(k)}$. When (4.12) and (4.13) are substituted into (4.11), all the surface integrals, except those contained in $H[\tau_{ij}]$, are found to cancel leaving

$$H[\tilde{\tau}_{ij}]=H[\tau_{ij}]-\int_{V_0}\frac{1}{2\mu}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2dV$$

$$-\sum_{l=1}^{N_L}\int_{V_L^{(l)}}\frac{1}{2\mu}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2dV. \tag{4.14}$$

In the reduction of (4.11) to (4.14), the surface integrals over S_2 which arise are:

$$2\int_{S_2}(hj_i\tilde{\tau}_{ij}n_j+bt_i\tilde{\tau}_{ij}n_j)\,dS-2\int_{S_2} u_i\tilde{\tau}_{ij}n_j dS$$

$$=2\int_{S_2}(hj_i\tilde{\tau}_{ij}n_j+bt_i\tilde{\tau}_{ij}n_j)\,dS$$

$$-2\int_{S_2}(u_qj_qj_i\tilde{\tau}_{ij}n_j+u_ql_ql_i\tilde{\tau}_{ij}n_j+u_qm_qm_i\tilde{\tau}_{ij}n_j)\,dS. \tag{4.15}$$

The sum of the integrals in (4.15) is zero because $\mathbf{u}$ satisfies (2.4a) and both τ_{ij} and $\tilde{\tau}_{ij}$ satisfy (2.4b) so that $\tilde{\tau}_{ij}n_jm_i=0$ on S_2. The cancellation of integrals over S_1, S_3 and S_4 follows similarly.

Surface integrals over $S_L^{(l)}$ arise from (4.12) and (4.13) which combine in (4.11) to give terms of the form

$$\int_{S_L^{(l)}} u_i \Delta\tilde{\tau}_{ij} n_j \, dS, \tag{4.16}$$

where $\Delta\tilde{\tau}_{ij}$ is the jump of $\tilde{\tau}_{ij}$ across the surfaces $S_L^{(l)}$. The tangential components of $\tau_{ij}n_j$ and $\bar{\tau}_{ij}n_j$ are continuous across $S_L^{(l)}$ since both satisfy (2.8). The normal components of $\tau_{ij}n_j$ and $\bar{\tau}_{ij}n_j$ take the same jump, as prescribed by (2.7). Hence $\Delta\tilde{\tau}_{ij}n_j$ is zero and (4.16) vanishes.

On the surfaces $S_K^{(k)}$ ($k = 1, \ldots, N_K$), the integrals arising from (4.12) are of the form

$$\int_{S_K^{(k)}} u_i \tilde{\tau}_{ij} n_j \, dS. \tag{4.17}$$

Since u_j has the form (3.9) on $S_K^{(k)}$ and both τ_{ij} and $\bar{\tau}_{ij}$ satisfy (2.9) and (2.10), it follows after substitution of (3.9) in (4.17) that (4.17) also vanishes.

In (4.14) the volume integrals involving $\tilde{\tau}_{ij}$ are positive unless $\tilde{\tau}_{ij}$ is zero or of the form $p_0\delta_{ij}$ where p_0 is a constant. Hence

$$H[\bar{\tau}_{ij}] \leqslant H[\tau_{ij}] \tag{4.18}$$

and the equality holds only if $\bar{\tau}_{ij} = \tau_{ij}$ or if $\bar{\tau}_{ij} = \tau_{ij} + p_0\delta_{ij}$. Theorem 2 follows from (4.18) and (4.2).

The constant p_0 will be zero if no uniform pressure field can satisfy the stress conditions (2.4*b*), (2.5*b*) and (2.6) when the given α, β, γ are replaced by zeros. In this case $D_e[u] = H[\bar{\tau}_{ij}]$ only if $\bar{\tau}_{ij} = \tau_{ij}$. This is the case, for example, if S_4 contains at least one point.

Theorems 1 and 2 contain the minimum and maximum principles given by Keller *et al.* (1967) as special cases in which the drops are of constant shape, S_2 is absent, and **j** is coincident with **n**.

The theorems 1 and 2 also apply to drops or regions of constant volume of one or more immiscible fluids in another fluid where the surface tensions are negligible ($\sigma^{(l)} = 0$). Then deformation of drops is to be expected in general.

5. Uniqueness theorem

THEOREM 3. *The solution* **u** *of a Stokes flow problem posed by* (2.1)–(2.13) *is unique to within a rigid body motion and the stress* τ_{ij} *is unique within a uniform pressure.*

Proof. Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be two solutions. Then (3.4) holds with $\mathbf{u} = \mathbf{u}^{(1)}$ and $\bar{\mathbf{u}} = \mathbf{u}^{(2)}$ and vice versa so the equality in (3.4) would hold. The first part of theorem 3 then follows from theorem 1.

Similarly, let $\tau_{ij}^{(1)}$ and $\tau_{ij}^{(2)}$ be the stresses corresponding to $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$. Then (4.18) holds with $\tau_{ij} = \tau_{ij}^{(1)}$ and $\bar{\tau}_{ij} = \tau_{ij}^{(2)}$ and vice versa so the equality would hold in (4.18). The second part of theorem 3 follows from theorem 2.

The arbitrary rigid body motion and the arbitrary uniform pressure implied in theorem 3 will be zero under the same conditions as discussed below (3.11) and (4.18).

Theorems 1, 2 and 3 can be applied to a single homogeneous fluid by deleting all references to suspended drops and particles.

The theorems also apply if any one, two, or three of the surfaces S_1, S_2, S_3, S_4 are absent. However, every point of S must be a point of one of the surfaces S_1, S_2, S_3, S_4. The boundary conditions on S which are permitted by (2.3)–(2.6) specify just enough components of velocity and traction to make the solution unique. This requires that sufficient components of velocity and/or traction be specified at each point of S that if $\tilde{u}_i$ and $\tilde{\tau}_{ij}$ are differences between two fields which both satisfy the boundary conditions on S, then the work rate $\tilde{u}_i\tilde{\tau}_{ij}n_j$ is zero at every point of S.

6. Infinite domains

In the theorems 1, 2 and 3, the domain V is assumed to be finite. The theorems can be applied to infinite domains if it is assumed that the velocity and stress fields decay fast enough so that the surface integrals which arise over a sphere at infinity vanish. The situation is similar to that of linear elastostatics for exterior domains treated by Gurtin & Sternberg (1961). As they point out, the rate at which a solution approaches specified values at infinity is an item of information which one would legitimately expect to infer from the solution, rather than a condition to be imposed on the solution in advance. A uniqueness theorem resting on an assumption of the rate of decay at infinity leaves in doubt the existence of solutions which approach the specified values at infinity less rapidly.

In the present section, generalizations of theorems 1, 2 and 3 are proved for infinite domains without assumptions of the rates of decay of the solutions at infinity. It is also shown that the comparison flows for the various theorems must be subject to a specification of the rate of dilation of the internal boundaries.

The nomenclature of § 2 will be used also for infinite domains with the understanding that the region V is now an exterior domain bounded internally by the surface S. The surface S is assumed to consist of a finite number of closed surfaces which lie within a finite sphere, $r = r_0$ where r_0 is a constant and r is the distance from the origin. The surface S is again considered in four parts S_1, S_2, S_3, S_4 according to the boundary conditions specified. It is assumed that the number of liquid drops, N_L, and the number of solid particles, N_K, in suspension in V are finite and that they also lie within the sphere $r = r_0$. The suspending fluid occupies the region V_0 which is the portion of V not occupied by solid particles or liquid drops. The surface S_0 of V_0 consists of S_1, S_2, S_3, S_4, $S_L^{(l)}$ $(l = 1, \ldots, N_L)$ and $S_K^{(k)}$ $(k = 1, \ldots, N_K)$.

The only boundary condition at infinity which will be considered is that the velocity approach a constant vector uniformly at infinity, i.e.

$$\lim_{r\to\infty} u_i = U_i, \tag{6.1}$$

where U_i is a given constant vector. Whenever the boundary condition (6.1) is imposed, a system of axes translating with velocity U_i may be used so that the condition (6.1) is replaced by

$$\lim_{r\to\infty} u_i = 0. \tag{6.2}$$

The condition (6.2) will be assumed to apply in all cases below.

An additional restriction that will be imposed for infinite domains is that the body force within V_0 be conservative, meaning there exists a single valued potential, $\Omega(\mathbf{x})$, such that

$$f_i = -\Omega_{,i}, \quad \mathbf{x} \quad \text{in} \quad V_0. \tag{6.3}$$

The boundary conditions on the interior boundaries S_1, S_2, S_3, S_4 are the same as for finite domains detailed by (2.3)-(2.6). It will be shown that for an infinite domain the total rate of expansion, θ^*, must also be specified for uniqueness of the solution. Hence the statement of the problem will be augmented by the requirement

$$-\int_S u_i n_i \, dS = \theta^*, \tag{6.4}$$

where θ^* may be a given function of time in general.

A complete statement of the problem considered in this section is to find $\mathbf{u}(\mathbf{x})$ in the infinite domain, V, described above satisfying (2.1)–(2.13), (6.2), (6.3) and (6.4).

The dissipation rate $D[\mathbf{u}]$ is again defined by (3.1) with the understanding that the integral over V_0 is now interpreted as the limit

$$\int_{V_0} 2\mu(e_{ij}[\mathbf{u}])^2 dV = \lim_{\rho\to\infty} \int_{V_{0\rho}} 2\mu(e_{ij}[\mathbf{u}])^2 dV. \tag{6.5}$$

where $V_{0\rho}$ is the portion of V_0 within a sphere $r = \rho$.

The excess dissipation rate $D_e^*[\mathbf{u}]$ for an infinite domain V is defined by

$$\begin{aligned} D_e^*[\mathbf{u}] = D[\mathbf{u}] &+ 2\int_{S_0} \Omega u_i n_i \, dS - 2\sum_{l=1}^{N_L} \int_{V_L^{(l)}} f_i u_i \, dV \\ &- 2\sum_{k=1}^{N_K} \int_{V_K^{(k)}} f_i u_i \, dV - 2\int_{S_2} u_i m_i \beta \, dS \\ &- 2\int_{S_3} (u_i t_i \alpha + u_i m_i \beta) \, dS - 2\int_{S_4} u_i \gamma_i \, dS + 2\sum_{l=1}^{N_L} \sigma^{(l)} \dot{A}^{(l)}. \end{aligned} \tag{6.6}$$

This definition (6.6) differs from (3.2) in that the rate of work done by body forces in V_0 has been replaced in (6.6) by the rate of change of potential energy due to the motion of the boundary S_0 of V_0. If the domain V_0 were finite, this potential energy term would be equal to the integral of $f_i u_i$ over V_0 by Gauss's theorem, (2.1) and (6.3). Then (6.6) would be equivalent to (3.2).

The counterpart of theorem 1 for infinite domains requires a representation theorem for u_i which is developed first below.

The velocity field $\mathbf{u}(\mathbf{x})$ is assumed to be continuous and to possess continuous derivatives up to second order within each of the domains V_0, $V_L^{(l)}$ $(l = 1, \ldots, N_L)$. At the boundaries of $V_L^{(l)}$ and $V_K^{(k)}$ the velocity $\mathbf{u}(\mathbf{x})$ is required to be continuous but its derivatives may be discontinuous. Then as shown in the appendix, u_i must be analytic within V_0 and $V_L^{(l)}$.

Equations (2.2) and (2.11) may be combined to give the usual equations of motion within V_0 and $V_L^{(l)}$.

$$u_{i,jj} - \frac{1}{\mu} p_{,i} + \frac{1}{\mu} f_i = 0. \tag{6.7}$$

Taking $(\partial/\partial x_i)\,\partial/\partial x_k$ of (6.7) and using (6.3) and (2.1) yields

$$p_{,iik}+\Omega_{,iik} = 0. \tag{6.8}$$

Taking $(\partial/\partial x_l)\,\partial/\partial x_l$ of (6.7) and using (6.8) shows that the velocity is biharmonic, i.e.

$$\nabla^4 u_i = 0. \tag{6.9}$$

A representation of biharmonic functions in an exterior domain has been developed by Gurtin & Sternberg (1961). A region $\mathscr{R}$ is defined as a deleted neighbourhood of infinity characterized by

$$r_0 < r < \infty, \tag{6.10}$$

where r_0 is a constant. For such a region they prove

THEOREM 4. *Let $F(r,\theta,\phi)$ be biharmonic in $\mathscr{R}$, where (r,θ,ϕ) are spherical polar co-ordinates. Then*

(a) $F(r,\theta,\phi)$ admits the representation

$$F(r,\theta,\phi) = \sum_{k=-\infty}^{\infty} h^{(k)}(r,\theta,\phi)+r^2\sum_{k=-\infty}^{\infty} H^{(k)}(r,\theta,\phi), \tag{6.11}$$

where $h^{(k)}(r,\theta,\phi)$ and $H^{(k)}(r,\theta,\phi)$ are solid harmonics of degree k and both infinite series are uniformly convergent in every closed subregion of $\mathscr{R}$;

(b) $F(r,\theta,\phi)$ in $\mathscr{R}$ has partial derivatives of all orders, series representations of which may be obtained by performing the corresponding termwise differentiations of (6.11), *the resulting expansions being also uniformly convergent in every closed subregion of $\mathscr{R}$;*

(c) if n is a fixed integer, the three statements

$$\text{(i)}\quad F(r,\theta,\phi) = O(r^{n-1}), \tag{6.12}$$

$$\text{(ii)}\quad F(r,\theta,\phi) = o(r^{n}), \tag{6.13}$$

$$\text{(iii)}\quad h^{(k)}(r,\theta,\phi) = H^{(k-2)}(r,\theta,\phi) = 0 \quad \text{for} \quad k \geqslant n \tag{6.14}$$

are equivalent and imply

$$\text{(iv)}\quad F_{,i}(r,\theta,\phi) = O(r^{n-2}). \tag{6.15}$$

The orders of magnitude $F = O(r^n)$ and $F = o(r^n)$ indicate, as usual, that $|r^{-n}F|$ remains bounded uniformly and $|r^{-n}F|$ approaches zero uniformly, respectively, as $r \to \infty$.

The following theorem follows from theorem 4.

THEOREM 5. *Suppose $u_i(x)$, $e_{ij}(x)$, $\tau_{ij}(x)$ and f_i in $\mathscr{R}$ satisfy* (2.1), (2.2), (2.11), (2.12) *and* (6.3). *Then if n is a fixed integer*

$$u_i(x) = o(r^n) \tag{6.16}$$

implies

$$\text{(i)}\quad u_i(x) = O(r^{n-1}), \tag{6.17}$$

$$\text{(ii)}\quad e_{ij} = O(r^{n-2}), \tag{6.18}$$

$$\text{(iii)}\quad p_{,i}+\Omega_{,i} = O(r^{n-3}). \tag{6.19}$$

Proof. Since $u_i(\mathbf{x})$ is biharmonic, theorem 4 applies with F replaced by u_i. Then (6.16), (6.13) and (6.12) imply (6.17). The definition (2.12) and (6.15) yield (6.18). Substituting (2.12), (2.11) and (6.3) into (2.2) and applying (6.15) again gives (6.19).

THEOREM 6. *A minimum principle for infinite domains. Let V be an exterior domain containing N_L liquid particles, N_K solid particles and internal boundaries S within a finite sphere $\mathbf{r} = r_0$. Let $\mathbf{u}(\mathbf{x})$ be a continuous solution of the Stokes flow problem satisfying* (2.1)–(2.13), (6.2), (6.3) *and* (6.4). *Let $\bar{\mathbf{u}}(\mathbf{x})$ be any continuous velocity field which is piecewise continuously differentiable and satisfies* (2.1), (2.3), (2.4*a*), (2.5*a*), (2.13), (6.4) *and*

$$\bar{u}_i = O(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{6.20}$$

Then

$$D_e^*[\mathbf{u}] \leqslant D_e^*[\bar{\mathbf{u}}], \tag{6.21}$$

where $D_e^*[\mathbf{u}]$ is defined by (6.6). *The equality in* (6.21) *holds only if $\bar{\mathbf{u}} = \mathbf{u}$.*

Proof. Let $\bar{\mathbf{u}} = \mathbf{u} + \tilde{\mathbf{u}}$. From (3.1) the forms (3.5) and (3.6) follow as before with the understanding that the integrals over V_0 are interpreted in the sense of (6.5). Replacing $\mathbf{u}$ by $\bar{\mathbf{u}}$ in (6.6) and using (3.6) and Gauss's theorem yields (6.22) below. In applying Gauss's theorem to V_0 in (3.6), the surface of V_0 is considered to consist of S_0 plus S_ρ where S_ρ is the surface of a sphere $r = \rho$, $\rho \to \infty$. Then

$$\begin{aligned} D_e^*[\bar{\mathbf{u}}] = {} & D[\mathbf{u}] + D[\tilde{\mathbf{u}}] + 2\int_{S_0} \tilde{u}_i \tau_{ij}^{(0)}[\mathbf{u}] n_j dS + 2\int_{V_0} f_i \tilde{u}_i dV \\ & - 2\sum_{l=1}^{N_L} \int_{S_L^{(l)}} \tilde{u}_i \tau_{ij}^{(l)}[\mathbf{u}] n_j dS + 2\sum_{l=1}^{N_L} \int_{V_L^{(l)}} f_i \tilde{u}_i dV \\ & + 2\int_{S_\rho} \tilde{u}_i \tau_{ij}[\mathbf{u}] n_j dS + 2\int_{S_0} \Omega(u_i + \tilde{u}_i) n_i dS \\ & - 2\sum_{l=1}^{N_L} \int_{V_L^{(l)}} f_i(u_i + \tilde{u}_i) dV - 2\sum_{k=1}^{N_K} \int_{V_K^{(k)}} f_i(u_i + \tilde{u}_i) dV \\ & - 2\int_{S_2} (u_i + \tilde{u}_i) m_i \beta dS - 2\int_{S_3} (u_i + \tilde{u}_i)(t_i \alpha + m_i \beta) dS \\ & - 2\int_{S_4} (u_i + \tilde{u}_i) \gamma_i dS + 2\sum_{l=1}^{N_L} \sigma^{(l)} (\dot{A}^{(l)} + \dot{\tilde{A}}^{(l)}), \end{aligned} \tag{6.22}$$

where the notation is the same as in (3.7). Since $\mathbf{u}$ and $\bar{\mathbf{u}}$ both satisfy (2.3), (2.4*a*) and (2.5*a*) and τ_{ij} satisfies (2.4*b*), (2.5*b*) and (2.6), the surface integrals over S_1, S_2, S_3 and S_4 involving $\tilde{\mathbf{u}}$ and τ_{ij} all cancel in (6.22). The surviving terms may be written

$$\begin{aligned} D_e^*[\bar{\mathbf{u}}] = {} & D_e^*[\mathbf{u}] + D[\tilde{\mathbf{u}}] + 2\sum_{l=1}^{N_L} \int_{S_L^{(l)}} \tilde{u}_i \Delta\tau_{ij} n_j dS + 2\int_{V_0} f_i \tilde{u}_i dV \\ & - 2\sum_{k=1}^{N_K} \int_{V_K^{(k)}} \tilde{u}_i f_i dV + 2\sum_{k=1}^{N_K} \int_{S_K^{(k)}} \tilde{u}_i \tau_{ij} n_j dS + 2\sum_{l=1}^{N_L} \sigma^{(l)} \dot{\tilde{A}}^{(l)} \\ & + 2\int_{S_\rho} \tilde{u}_i \tau_{ij}[\mathbf{u}] n_j dS + 2\int_{S_0} \Omega \tilde{u}_i n_i dS. \end{aligned} \tag{6.23}$$

Equation (6.23) is the counterpart of (3.8). The terms in (6.23) which are summed over l and k pertain to the liquid and solid particles and add up to zero as shown below (3.8). Applying Gauss's theorem to the region V_0 considered bounded by S_0 internally and S_ρ externally and using (6.3) and (2.1) yields

$$\int_{S_0} \Omega \tilde{u}_i n_i dS = -\int_{V_0} f_i \tilde{u}_i dV - \int_{S_\rho} \Omega \tilde{u}_i n_i dS. \tag{6.24}$$

Substituting (6.24) in (6.23) gives

$$D_e^*[\bar{\mathbf{u}}] = D_e^*[\mathbf{u}] + D[\tilde{\mathbf{u}}] + 2\int_{S_\rho} \tilde{u}_i(\tau_{ij}[\mathbf{u}] - \delta_{ij}\Omega) n_j dS. \tag{6.25}$$

Using (2.11) the integral over S_ρ in (6.25) is

$$\int_{S_\rho} \tilde{u}_i(\tau_{ij}[\mathbf{u}] - \delta_{ij}\Omega) n_j dS = \int_{S_\rho} \mu \tilde{u}_i e_{ij}[\mathbf{u}] n_j dS - \int_{S_\rho} (p+\Omega)\tilde{u}_i n_i dS. \tag{6.26}$$

Theorem 5 applies to u_i with $n = 0$ by virtue of (6.2). Hence $e_{ij}[\mathbf{u}] = O(r^{-2})$ by (6.18). Further, $\tilde{u}_i = O(r^{-1})$ by (6.20) and (6.17). It follows that the first integral on the right of (6.26) is zero in the limit $\rho \to \infty$.

If the integration of (6.19) is considered along a path lying on the sphere S_ρ, it follows that on $r = \rho$

$$p + \Omega = p^* + F(r, \theta, \phi), \tag{6.27}$$

where p^* is a constant and $F(r, \theta, \phi)$ is a function of order $O(r^{-2})$. Hence

$$\int_{S_\rho} (p+\Omega)\tilde{u}_i n_i dS = p^* \int_{S_\rho} \tilde{u}_i n_i dS + \int_{S_\rho} F(\rho, \theta, \phi)\tilde{u}_i n_i dS. \tag{6.28}$$

The first integral on the right of (6.28) is zero since $\mathbf{u}$ and $\bar{\mathbf{u}}$ satisfy (6.4) and the second integral is zero in the limit $\rho \to \infty$. Hence (6.28) and (6.26) are zero and theorem 6 follows from (6.25). In the present case, u_i and $\bar{u}_i$ cannot differ by a rigid body motion because of the boundary condition at infinity so the equality holds in (6.21) only if $\bar{\mathbf{u}} = \mathbf{u}$.

A maximum principle for infinite domains corresponding to theorem 2 for finite domains can be derived if the excess power is redefined for infinite domains as follows

$$\begin{aligned} H^*[\tau_{ij}] = {} & 2\int_{S_1} g_i \tau_{ij} n_j dS + 2\int_{S_2} (hj_i \tau_{ij} n_j + bt_i \tau_{ij} n_j) dS \\ & + 2\int_{S_3} hj_i \tau_{ij} n_j dS - 2\theta^* p^* \\ & - \int_{V_0} \frac{1}{2\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV \\ & - \sum_{l=1}^{N_L} \int_{V_L^{(l)}} \frac{1}{2\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV. \end{aligned} \tag{6.29}$$

The difference between (6.29) and (4.1) is that the term $-2\theta^* p^*$ has been added in (6.29). The total rate of expansion, θ^*, defined by (6.4) is a part of the given kinematic data and the work done by the pressure and body forces at infinity represented by p^* is therefore included in the excess power. $H_e^*[\tau_{ij}]$ is defined only when p^* exists as defined by

$$p^* = \lim_{r\to\infty} (p+\Omega) = \lim_{r\to\infty} (-\tfrac{1}{3}\tau_{kk}+\Omega). \tag{6.30}$$

When τ_{ij} is the stress tensor corresponding to a solution of (2.1)-(2.13), (6.2), (6.3) and (6.4), then

$$H^*[\tau_{ij}] = D_e^*[\mathbf{u}], \tag{6.31}$$

where $D_e^*[\mathbf{u}]$ is given by (6.6). To prove (6.31), we proceed as in proving (4.2). In the present case, (4.3), (4.4) and (4.5) hold also. Using (4.3)-(4.5) in (6.29) and subtracting twice the integral of $u_i\tau_{ij}n_j$ over S_4 gives

$$\begin{aligned} H^*[\tau_{ij}] = {} & 2\int_S u_i\tau_{ij}n_j dS - 2\int_{S_2} u_i m_i \beta dS \\ & -2\int_{S_3} (u_i t_i \alpha + u_i m_i \beta) dS - 2\int_{S_4} u_i \gamma_i dS \\ & -2\theta^* p^* - D[\mathbf{u}]. \end{aligned} \tag{6.32}$$

Instead of (4.7), the conservation of energy now takes the form

$$\int_S u_i\tau_{ij}n_j dS + \int_{S_\rho} u_i\tau_{ij}n_j dS + \int_V f_i u_i dV = D[\mathbf{u}] + \sum_{l=1}^{N_L} \sigma^{(l)}\dot{A}^{(l)}, \tag{6.33}$$

where the integrals over S_ρ and V are interpreted as the limits for $\rho\to\infty$. Using (6.3) and Gauss's theorem these terms may be written

$$\begin{aligned} \int_{S_\rho} u_i\tau_{ij}n_j dS + \int_V f_i u_i dV = {} & -\int_{S_\rho} u_i p n_i dS - \int_{S_\rho} \Omega u_i n_i dS \\ & -\int_{S_0} \Omega u_i n_i dS + \sum_{l=1}^{N_L}\int_{V_L^{(l)}} f_i u_i dV + \sum_{k=1}^{N_K}\int_{V_K^{(k)}} f_i u_i dV. \end{aligned} \tag{6.34}$$

The two integrals over S_ρ on the right of (6.34) may be replaced by $-\theta^* p^*$ in view of (6.27) and the fact that $u_i = O(r^{-1})$. Substituting (6.34) into (6.33) and using (6.33) to eliminate the integral over S in (6.32) yields $H^*[\tau_{ij}]$ in a form identical to (6.6) so (6.31) is proved.

THEOREM 7. *A maximum principle for infinite domains. Let V be an exterior domain containing N_L liquid particles, N_K solid particles and internal boundaries S within a finite sphere $r = r_0$. Let $\mathbf{u}(\mathbf{x})$ be a continuous solution of the Stokes flow problem satisfying* (2.1)–(2.13), (6.2), (6.3), (6.4). *Let $\bar{\tau}_{ij}$ be any stress tensor defined in V_0 and $V_L^{(l)}$ which is piecewise continuous and piecewise continuously differentiable and satisfies* (2.2), (2.4*b*), (2.5*b*), (2.6)–(2.10). *On surfaces of discontinuity of $\bar{\tau}_{ij}$ the traction $n_i'\bar{\tau}_{ij}$ is required to be continuous where n_i' is the normal to the surface of the discontinuity of $\bar{\tau}_{ij}$. Further, the limit, $\bar{p}^*$, defined by* (6.30) *must exist and*

$$\bar{\tau}_{ij} - \tfrac{1}{3}\bar{\tau}_{kk}\delta_{ij} = O(r^{-2}) \quad \text{as} \quad r\to\infty. \tag{6.35}$$

Then $$D_e^*[\mathbf{u}] \geqslant H^*[\bar{\tau}_{ij}]. \tag{6.36}$$

The equality in (6.36) holds only if $\bar{\tau}_{ij} = \tau_{ij}$ or $\bar{\tau}_{ij} = \tau_{ij}+p_0\delta_{ij}$ where p_0 is a constant.

Proof. Let $\bar{\tau} = \tau_{ij}+\tilde{\tau}_{ij}$, where τ_{ij} is the stress tensor corresponding to the solution $\mathbf{u}$. In (6.36), $H^*[\bar{\tau}_{ij}]$ is given by (6.29) with $[\tau_{ij}]$ replaced by $[\bar{\tau}_{ij}]$ and p^* replaced by $\bar{p}^*$. The same steps that were used to convert (4.9) to (4.11) yield

$$\begin{aligned}H^*[\bar{\tau}_{ij}] = H^*[\tau_{ij}] &- \int_{V_0}\frac{1}{2\mu}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ &- \sum_{l=1}^{N_L}\int_{V_L^{(l)}}\frac{1}{2\mu}(\tau_{ij}-\tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV \\ &- 2\int_{V_0} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})\,dV \\ &- 2\sum_{l=1}^{N_L}\int_{V_L^{(l)}} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})dV \\ &+ 2\int_{S_1} g_i\tilde{\tau}_{ij}n_j dS + 2\int_{S_2}(hj_i\tilde{\tau}_{ij}n_j + bt_i\tilde{\tau}_{ij}n_j)\,dS \\ &+ 2\int_{S_3} hj_i\tilde{\tau}_{ij}n_j dS - 2\theta^*\tilde{p}^*,\end{aligned} \tag{6.37}$$

where $\tilde{p}^* = \bar{p}^* - p^*$. The integral over V_0 containing e_{ij} in (6.37) may be rewritten by the same steps as in (4.12) to yield

$$\int_{V_0} e_{ij}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tau_{kk}\delta_{ij})\,dV = \int_{S_0} u_i\tilde{\tau}_{ij}n_j dS + \int_{S_\rho} u_i\tilde{\tau}_{ij}n_j dS. \tag{6.38}$$

The integral over S_ρ in (6.38) may be written

$$\int_{S_\rho} u_i\tilde{\tau}_{ij}n_j dS = \int_{S_\rho} u_i(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})n_j dS - \int_{S_\rho} u_i\tilde{p}^* n_i d\tau. \tag{6.39}$$

The first integral on the right of (6.39) is zero in the limit $\rho\to\infty$ due to (6.2) and (6.35); the second integral is equal to $-2\theta^*\tilde{p}^*$. Substituting (6.39) and (6.38) into (6.37) and using the same arguments as used in connexion with (4.14) gives

$$\begin{aligned}H^*[\bar{\tau}_{ij}] = H^*[\tau_{ij}] &- \int_{V_0}\frac{1}{2\mu}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ &- \sum_{l=1}^{N_L}\int_{V_L^{(l)}}\frac{1}{2\mu}(\tilde{\tau}_{ij}-\tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV.\end{aligned} \tag{6.40}$$

The integrals in (6.40) are positive unless $\tilde{\tau}_{ij}$ is zero or of the form $p_0\delta_{ij}$ where p_0 is a constant throughout V_0 and $V_L^{(l)}$. Hence

$$H^*[\bar{\tau}_{ij}] \leqslant H^*[\tau_{ij}] \tag{6.41}$$

and the equality holds only if $\bar{\tau}_{ij} = \tau_{ij}$ or if $\bar{\tau}_{ij} = \tau_{ij}+p_0\delta_{ij}$. Theorem 7 follows from (6.41) and (6.31). The constant p_0 will be zero under the same conditions discussed below (4.18).

THEOREM 8. *Uniqueness theorem for infinite domains. Let V be an exterior domain containing N_L liquid particles, N_K solid particles and internal boundaries S within a finite sphere $r = r_0$. Then the solution* $\mathbf{u}$ *of a Stokes flow problem posed by* (2.1)–(2.13), (6.2), (6.3) *and* (6.4) *is unique and the stress τ_{ij} is unique to within a uniform pressure.*

Proof. The proof follows from theorems 6 and 7 by the same arguments by which theorem 3 follows from theorems 1 and 2.

In theorems 6, 7 and 8 the requirement that θ^* be specified as part of the given data may be redundant if sufficient velocity components are specified by (2.3)–(2.5) to compute the integral in (6.4). In this case, the separate requirement (6.4) may be deleted.

The physical significance of specifying θ^* is illustrated by the following simple problem.

A hollow spherical cavity of radius r_0, centred at the origin, is surrounded by a uniform viscous liquid extending to infinity. Suppose the body forces are zero and the internal pressure in the cavity is p_0, a given constant. Find the creeping motion of the fluid.

The solution of this problem is

$$u_r = (p_0 - c_1)\, r_0^2 / 4\mu r^2, \tag{6.42}$$

which is not unique because c_1 is an arbitrary constant equal to the pressure at infinity which was not specified.

If the problem is augmented by requiring θ^* to be a given value, the solution is

$$u_r = \theta^* / 4\pi r^2, \tag{6.43}$$

which is unique. The stress tensor is now also unique. In effect, specifying θ^* determines the pressure at infinity.

7. Spatially periodic flows

Consider an infinite pipe whose cross-section is variable, but periodic with respect to a co-ordinate x_1 with periodicity λ. The walls of the pipe are fixed and rigid and may contain additional internal boundaries provided they are also fixed and rigid. Let the remaining space be filled with a viscous liquid containing liquid drops and solid particles which are also distributed periodically in x_1. Body forces f_i are assumed to be periodic in x_1 also. It is assumed that the velocity field of any Stokes flow in the pipe under these conditions is periodic in x_1 and consists of a series of identical cells.

Each cell has two identical surfaces, say S_a and S_b in order of increasing x_1, spaced λ apart. S_a and S_b need not be plane, but are chosen to extend entirely across the flow and not to intersect any liquid drops or solid particles. The remaining surface of the cell, say S_c, consists entirely of fixed boundaries. Hence

$$u_i = 0 \quad \text{on} \quad S_c. \tag{7.1}$$

Let the volume of a typical flow cell be V with boundary S equal to the sum of S_a, S_b and S_c. Let V contain N_L liquid drops and N_K rigid, solid particles. Let V_0, $V_L^{(l)}$, $V_K^{(k)}$ be the parts of V occupied by suspending fluid, liquid drops and solid particles respectively with surfaces S_0, $S_L^{(l)}$ and $S_K^{(k)}$.

The discharge, Q, through the pipe must be the same for all cross-sections, i.e.

$$\int_{S'} u_n dS = Q \quad \text{all } S', \tag{7.2}$$

where S' is any cross-section of the flow cell and u_n is the component of velocity normal to S'. The discharge Q includes suspending fluid, liquid drops and solid particles. The general problem considered is to find $\mathbf{u}(\mathbf{x})$ in V satisfying (2.1), (2.2), (2.7)–(2.13), (7.1) and (7.2) with Q given.

Substituting (2.11) in (2.2), it may be seen that since f_i and u_i are periodic, $p_{,i}$ is periodic in x_1 and $\partial p/\partial s$ is identical for corresponding paths on S_a and S_b. Then by integrating along S_a and S_b it follows that any difference of pressures at corresponding points of S_a and S_b is the same constant, say Δp, for all pairs of corresponding points. A mean pressure gradient, p_x, is defined by

$$p_x = \Delta p/\lambda. \tag{7.3}$$

The dissipation $D[\mathbf{u}]$ in V is given by (3.1). The excess dissipation $D_e'[\mathbf{u}]$ for the present case is defined by

$$D_e'[\mathbf{u}] = D[\mathbf{u}] - 2\int_V f_i u_i dV + 2\sum_{l=1}^{N_L} \sigma^{(l)} \dot{A}^{(l)}, \tag{7.4}$$

where $\dot{A}^{(l)}$ is given by (3.3).

THEOREM 9. *A minimum principle. Let* $\mathbf{u}(\mathbf{x})$ *be a continuous solution of a periodic Stokes flow problem satisfying* (2.1), (2.2), (2.7)–(2.13), (7.1) *and* (7.2). *Let* $\bar{\mathbf{u}}(\mathbf{x})$ *be any continuous periodic velocity field which is piecewise continuously differentiable and satisfies* (2.1), (2.13), (7.1) *and* (7.2). *Then*

$$D_e'[\mathbf{u}] \leqslant D_e'[\bar{\mathbf{u}}]. \tag{7.5}$$

The equality holds only if $\mathbf{u} = \bar{\mathbf{u}}$.

Proof. Let $\bar{\mathbf{u}} = \mathbf{u} + \tilde{\mathbf{u}}$. Then (3.5) and (3.6) apply in the present case also. Replacing $\mathbf{u}$ by $\bar{\mathbf{u}}$ in (7.4), using (3.6) and Gauss's theorem yields

$$\begin{aligned} D_e'[\bar{\mathbf{u}}] = D[\mathbf{u}] + D[\tilde{\mathbf{u}}] + 2\int_{S_0} \tilde{u}_i \tau_{ij}^{(0)}[\mathbf{u}] n_j dS + 2\int_{V_0} f_i \tilde{u}_i dV \\ -2\sum_{l=1}^{N_L}\int_{S_L^{(l)}} \tilde{u}_i \tau_{ij}^{(l)}[\mathbf{u}] n_j dS + 2\sum_{l=1}^{N_L}\int_{V_L^{(l)}} f_i \tilde{u}_i dV \\ -2\int_V f_i(u_i + \tilde{u}_i) dV + 2\sum_{l=1}^{N_L} \sigma^{(l)}(\dot{A}^{(l)} + \dot{\tilde{A}}^{(l)}), \end{aligned} \tag{7.6}$$

where the notation is the same as in (3.7) except that in (7.6) S_0 is the sum of S_a, S_b, S_c, $S_L^{(l)}$ and $S_K^{(k)}$. The portion of the integral over S_0 in (7.6) associated with

$S_L^{(l)}$ and $S_K^{(k)}$ combines to nullify the same terms as in (3.7); the portion over S_c is zero by (7.1). This leaves only the integrals over S_a and S_b which may be written

$$2\int_{S_a} \tilde{u}_i \tau_{ij}[\mathbf{u}] n_j dS + 2\int_{S_b} \tilde{u}_i \tau_{ij}[\mathbf{u}] n_j dS$$

$$= 2\int_{S_b} \tilde{u}_i [\tau_{ij}^{(b)}[\mathbf{u}] - \tau_{ij}^{(a)}[\mathbf{u}]] n_j dS. \tag{7.7}$$

The difference of the stress tensors on S_b and S_a represented by $\tau_{ij}^{(b)} - \tau_{ij}^{(a)}$ in (7.7) is equal to $\Delta p \delta_{ij}$ at every point. Further, the integral of $\tilde{u}_i n_i$ is zero over S_b because $\tilde{\mathbf{u}}$ and $\mathbf{u}$ satisfy (7.2). Hence (7.7) is zero and (7.6) reduces to

$$D_e'[\tilde{\mathbf{u}}] = D_e'[\mathbf{u}] + D[\tilde{\mathbf{u}}]. \tag{7.8}$$

Then (7.5) follows from (7.8).

A maximum principle similar to theorem 2 for spatially periodic flows can be derived for suitably restricted comparison stress fields, $\overline{\tau}_{ij}$. The stress deviator of $\overline{\tau}_{ij}$ is required to be periodic in x_1 and the pressure $\overline{p}$ must exhibit a constant difference $\Delta\overline{p}$ for all pairs of corresponding points on S_a and S_b of the typical cell. Thus

$$\overline{\tau}_{ij} - \tfrac{1}{3}\overline{\tau}_{kk}\delta_{ij} = \text{periodic in } x_1, \tag{7.9}$$

$$[-\tfrac{1}{3}\overline{\tau}_{kk}]_A - [-\tfrac{1}{3}\overline{\tau}_{kk}]_B = \Delta\overline{p}, \tag{7.10}$$

where $\Delta\overline{p}$ is a constant and A and B are any pair of corresponding points on S_a and S_b.

The excess power $H'[\tau_{ij}]$ is defined for a periodic Stokes flow having a discharge Q and any stress field τ_{ij} satisfying (7.10) by

$$H'[\tau_{ij}] = 2Q\Delta p - \int_{V_0} \frac{1}{2\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV$$

$$- \sum_{l=1}^{N_L} \int_{V_L^{(l)}} \frac{1}{2\mu}(\tau_{ij} - \tfrac{1}{3}\tau_{kk}\delta_{ij})^2 dV, \tag{7.11}$$

where V_0 and $V_L^{(l)}$ refer to the typical cell of the flow.

When the stress tensor τ_{ij} and concomitant pressure drop Δp are those of a solution $\mathbf{u}(\mathbf{x})$ of the periodic Stokes flow problem with discharge Q, then

$$H'[\tau_{ij}] = D_e'[\mathbf{u}]. \tag{7.12}$$

To prove (7.12), we use (4.7) to show that

$$Q\Delta p = D[\mathbf{u}] + \sum_{l=1}^{N_L} \sigma^{(l)} \dot{A}^{(l)} - \int_V f_i u_i dV. \tag{7.13}$$

Substituting (7.13) in (7.11) and identifying terms with (7.4) yields (7.12).

THEOREM 10. *A maximum principle. Let* $\mathbf{u}(\mathbf{x})$ *be a continuous solution of a periodic Stokes flow problem satisfying* (2.1), (2.2), (2.7)-(2.13), (7.1) *and* (7.2). *Let* $\overline{\tau}_{ij}$ *be any stress tensor defined in* V_0 *and* $V_L^{(l)}$ *which is piecewise continuous* *and piecewise continuously differentiable and satisfies* (2.2), (2.7)–(2.10), (7.9) *and* (7.10). *On surfaces of discontinuity of* $\bar{\tau}_{ij}$ *the traction* $n_i'\bar{\tau}_{ij}$ *is required to be continuous. Then*

$$D_e'[u] \geqslant H'[\bar{\tau}_{ij}]. \tag{7.14}$$

The equality holds only if $\bar{\tau}_{ij} = \tau_{ij}$ or $\bar{\tau}_{ij} = \tau_{ij} + p_0 \delta_{ij}$ where p_0 is a constant.

Proof. Let $\bar{\tau}_{ij} = \tau_{ij} + \tilde{\tau}_{ij}$. Substituting $\bar{\tau}_{ij}$ in (7.11), using (4.10) and collecting terms as in (4.11) yields

$$\begin{aligned} H'[\bar{\tau}_{ij}] = H'[\tau_{ij}] &- \int_{V_0} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ &- \sum_{l=1}^{N_L} \int_{V_L^{(l)}} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ &- 2\int_{V_0} e_{ij}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij}) dV \\ &- 2\sum_{l=1}^{N_L} \int_{V_L^{(l)}} e_{ij}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij}) dV \\ &+ 2Q\Delta\tilde{p}, \end{aligned} \tag{7.15}$$

where $\Delta\tilde{p}$ is defined by (7.10) with $\bar{\tau}_{ij}$ replaced by $\tilde{\tau}_{ij}$. Equations (4.12) and (4.13) apply in the present case with S_0 equal to the sum of S_a, S_b, S_c, $S_L^{(l)}$ and $S_K^{(k)}$. Considering the fact that $\tilde{\tau}_{ij}$ satisfies conditions of the form (7.9) and (7.10), it is found that after substitution of (4.12) and (4.13) in (7.15) that (7.15) can be reduced to

$$\begin{aligned} H'[\bar{\tau}_{ij}] = H'[\tau_{ij}] &- \int_{V_0} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV \\ &- \sum_{l=1}^{N_L} \int_{V_L^{(l)}} \frac{1}{2\mu}(\tilde{\tau}_{ij} - \tfrac{1}{3}\tilde{\tau}_{kk}\delta_{ij})^2 dV. \end{aligned} \tag{7.16}$$

Theorem 10 follows from (7.16) and (7.12).

A uniqueness theorem for periodic Stokes flow can be derived from theorems 9 and 10 by the same arguments used to prove theorem 3. The result is

THEOREM 11. *Uniqueness theorem for periodic flows. A periodic solution* $\mathbf{u}(\mathbf{x})$ *of a periodic Stokes flow problem satisfying* (2.1), (2.2), (2.7)–(2.13), (7.1) *and* (7.2) *is unique for a given discharge Q and the stress* τ_{ij} *is unique to within a uniform pressure.*

If f_i is conservative so that it has a potential Ω and the solid particles and liquid drops are neutrally buoyant, theorems 9–11 can be simplified.

The condition that the suspended drops and particles be neutrally buoyant particles is equivalent to the requirement that Ω be continuous in V. Since f_i is assumed to be periodic in x_1, any difference of Ω at corresponding points of S_a and S_b is a constant, i.e.

$$[\Omega]_A - [\Omega]_B = \Delta\Omega, \tag{7.17}$$

where $\Delta\Omega$ is a constant and A and B are any pair of corresponding points on S_a and S_b. It follows that

$$\int_V u_i f_i dV = \int_V \bar{u}_i f_i \, dV, \tag{7.18}$$

where u_i and $\bar{u}_i$ are the velocity fields in theorem 9. Proof of (7.18) follows by use of Gauss's theorem, (7.1), (7.2) and (7.17). Adding twice (7.18) to (7.5) yields

THEOREM 12. *If the suspended liquid drops and solid particles are neutrally buoyant and the body forces are conservative, then theorem* 9 *holds with* (7.5) *replaced by*

$$D[\mathbf{u}]+2\sum_{l=1}^{N_L}\sigma^{(l)}\dot{A}^{(l)} \leqslant D[\bar{\mathbf{u}}]+2\sum_{l=1}^{N_L}\sigma^{(l)}\dot{\bar{A}}^{(l)}. \tag{7.19}$$

Similarly, theorem 10 may be replaced by

THEOREM 13. *If the suspended liquid drops and solid particles are neutrally buoyant and the body forces are conservative, then theorem* 10 *holds with* (7.14) *replaced by*

$$D[\mathbf{u}]+2\sum_{l=1}^{N_L}\sigma^{(l)}\dot{A}^{(l)} \geqslant H'[\bar{\tau}_{ij}]+2Q\Delta\Omega. \tag{7.20}$$

The uniqueness theorem for periodic Stokes flows, theorem 11, remains unchanged whether the suspended drops and particles are neutrally buoyant or not.

If there are no liquid drops present, or if the shape of the liquid drops is assumed to be constant, the terms involving $\sigma^{(l)}$ in (7.19) and (7.20) do not appear and theorems 12 and 13 give bounds on the dissipation $D[\mathbf{u}]$ directly. In this case theorem 9 can be reformulated as follows:

THEOREM 14. *The solution* $\mathbf{u}(\mathbf{x})$ *of a periodic Stokes flow problem satisfying* (2.1), (2.2), (2.7)–(2.13), (7.1) *and* (7.2) *produces less dissipation than any other periodic flow* $\bar{\mathbf{u}}(\mathbf{x})$ *satisfying* (2.1), (7.1) *and* (7.2) *for the same discharge* Q *provided* (i) $\bar{\mathbf{u}}(\mathbf{x})$ *is continuous and piecewise continuously differentiable*; (ii) *body forces are conservative*; (iii) *suspended solid particles and liquid drops are neutrally buoyant and of constant shape.*

Theorem 14 can be applied to the steady laminar flow of a uniform liquid with no suspended particles in an infinite pipe of any uniform cylindrical cross-section. Such a flow may be considered periodic with any periodicity λ, $0 < \lambda < \infty$. Then theorem 14 states that the laminar flow solution of this problem has less dissipation than any spatially periodic comparison flow of the same discharge. This is a result that was proved previously by Thomas (1942) for the case of uniform flow in a circular pipe.

Theorem 14 is also of interest for approximate computation of the pressure drop in a model of capillary blood flow in which the red blood cells are represented as deformed liquid drops of constant shape spaced periodically in a uniform circular tube.

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Appendix. Analyticity of u_i

The equations of motion (6.7) and continuity (2.1) may be written

$$\nabla^2 u_i = \frac{\partial F}{\partial x_i} \tag{A 1}$$

and $$u_{i,i} = 0, \tag{A2}$$

where $$F = \frac{1}{\mu}(p+\Omega). \tag{A3}$$

The equations (A 1) and (A 2) are identical in form to the equations of linear elasticity with Poisson's ratio equal to $\frac{1}{2}$ and zero body forces, as discussed by Duffin (1956). Assuming only that the derivatives in (A 1) and (A 2) exist and are continuous in an open domain E, Duffin (1956) proves that F is harmonic, i.e. $\nabla^2 F = 0$ and hence F is analytic in E. Now (A 1) may be regarded as Poisson's equation on u_i where $F_{,i}$ is analytic. The differentiability theorem given by Courant & Hilbert (1962, p. 345) for a general second-order elliptic equation then ensures that u_i is also analytic. Thus F and u_i possess derivatives of all orders.

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